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AN ARBITRARILY ORIENTED CRACK IN A SEMI-INFINITE MEDIUM WITH A SURFACE LAYER UNDER TENSION

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Abstract—The title problem is analyzed by the use of the continuous distribution of dislocations in the framework of plane elastostatics. A set of singular integral equations for the dislocation density is readily set up, whose solution is obtained in the form of the product of the series of Chebyshev polynomials of the first kind and their weight function. The variation of the stress intensity factor due to the geometrical configuration and the material combination is shown in graphs. The probable angle of crack growth due to Erdogan–Sih criterion for brittle materials is also shown.

1. INTRODUCTION

The structural strength of composite materials is dependent to a considerable extent on the size, shape and orientation of flaws which exist in the medium. Thus in designing with the composite materials, it is necessary to have a good estimate for the stress state caused by these flaws, which may generally be idealized as plane cracks.

Recently the singular stress field around a crack located near a bi-material interface were evaluated in a series of papers [1–6]. The elastic interaction of a surface layer and a crack in a semi-infinite medium under tension has been studied under the simplification that the surface layer is supposed as a membrane [7] or a beam [8] and the crack is perpendicular to the surface.

The primary interest of this paper is in the evaluation of the disturbed stress field near a crack in a semi-infinite medium with a surface layer and transmitting a tensile load parallel to the surface. We will assume that the location and orientation of the crack with respect to the interface is arbitrary and the problem is one of plane elastostatic states. Further the surface layer is supposed to be an elastic strip of constant width. It may be noted that the companion problem of longitudinal shear loading has been discussed recently [9].

In order to apply the approach of the continuous arrays of dislocation to the aforementioned problem, we first derive the complex potentials expressing an isolated edge dislocation in the composite medium. By replacing the crack by the dislocation arrays, we can readily set up a system of singular integral equations for dislocation density. Applying the technique developed by Erdogan [10], we are able to get the solution, which preserves the essential features of the singular stress field near the crack and the stress intensity factors are easily evaluated.

Numerical calculations are also carried out and the variation of the stress intensity factor due to the geometrical configuration and the material combination is shown in graphs. The probable angle of crack growth based on Erdogan–Sih criterion [11] for brittle materials is also shown.

2. STATEMENT OF PROBLEM

Let the homogeneous isotropic elastic medium m occupy the lower half $y_1 < 0$ of $z_1 = x_1 + iy_1$ plane and the elastic layer s of uniform width h be perfectly bonded to the straight edge $y_1 = 0$. The semi-infinite medium contains a line crack L of length $2a$, forming an angle α with x_1 axis, see Fig. 1. The center O of the crack is located at the point $x_1 = 0$, $y_1 = -d$. We also use the complex coordinates $z = x + iy$ having the origin at the crack center, in which x axis lies along the crack L. The relation between these two coordinates is given by

$$z = (z_1 + id)e^{-i\alpha}. \tag{1}$$

It will be assumed that the composite medium is subjected to a tensile load along x_1 axis at infinity,

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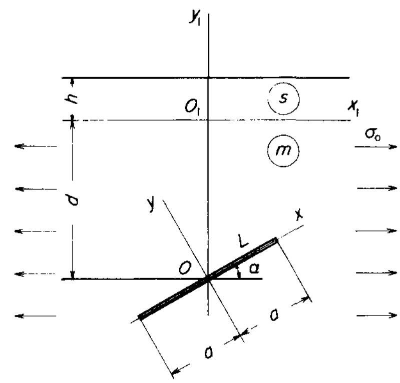

Fig. 1. Configuration and coordinate system.

the surfaces of the surface layer and the crack being free from tractions. Denoting complex potentials of the semi-infinite medium and the surface layer by $\Phi_{m_1}(z_1)$, $\Psi_{m_1}(z_1)$ and $\Phi_{s_1}(z_1)$, $\Psi_{s_1}(z_1)$, respectively, we can write the boundary conditions of the present problem as:

(a) At infinity $|x_1|\to\infty$,

$$\Phi_{m_1}(z_1)\to\sigma_0/4,\quad \Psi_{m_1}(z_1)\to-\sigma_0/2,$$
$$\Phi_{s_1}(z_1)\to(\delta+\eta-1)\sigma_0/4,\quad \Psi_{s_1}(z_1)\to-(\delta+\eta-1)\sigma_0/2, \tag{2}$$

where

$$\delta=\frac{\Gamma\kappa_m+1}{\kappa_s+1},\quad \eta=\frac{\Gamma+\kappa_s}{\kappa_s+1},\quad \Gamma=\frac{\mu_s}{\mu_m}, \tag{3}$$

μ being the shear modulus and κ the equivalent Poisson's ratio ($\kappa=3-4\sigma$ for plane strain and $=(3-\sigma)/(1+\sigma)$ for generalized plane stress). The subscripts m and s refer to the semi-infinite medium and the surface layer, respectively.

(b) Along the surface $z_1=x_1+ih$,

$$\Phi_{s_1}(z_1)+\overline{\Phi_{s_1}(z_1)}+z_1\overline{\Phi'_{s_1}(z_1)}+\overline{\Psi_{s_1}(z_1)}=0. \tag{4}$$

(c) Along the interface $z_1=x_1$,

$$\Phi_{m_1}(z_1)+\overline{\Phi_{m1}(z_1)}+z_1\overline{\Phi'_{m1}(z_1)}+\overline{\Psi_{m_1}(z_1)}=\Phi_{s_1}(z_1)+\overline{\Phi_{s_1}(z_1)}+z_1\overline{\Phi'_{s_1}(z_1)}+\overline{\Psi_{s_1}(z_1)} \tag{5}$$

$$[\kappa_m\Phi_{m_1}(z_1)-\overline{\Phi_{m_1}(z_1)}-z_1\overline{\Phi'_{m1}(z_1)}-\overline{\Psi_{m_1}(z_1)}]/2\mu_m=[\kappa_s\Phi_{s_1}(z_1)-\overline{\Phi_{s_1}(z_1)}-z_1\overline{\Phi'_{s_1}(z_1)}-\overline{\Psi_{s_1}(z_1)}]/2\mu_s. \tag{6}$$

(d) On the rim of the crack $z=x$, $(|x|<a)$,

$$\Phi_m(z)+\overline{\Phi_m(z)}+z\overline{\Phi'_m(z)}+\overline{\Psi_m(z)}=0, \tag{7}$$

where $\Phi_m(z)$ and $\Psi_m(z)$ are complex potentials for the semi-infinite medium in terms of z and by the use of eqn (1) they can be obtained by[12]

$$\Phi_m(z)=\Phi_{m_1}(ze^{i\alpha}-id),$$
$$\Psi_m(z)=\Psi_{m_1}(ze^{i\alpha}-id)e^{i2\alpha}+id\Phi'_m(ze^{i\alpha}-id)e^{i2\alpha}. \tag{8}$$

Using the conventional superposition technique, we will look for the complex potentials which satisfy all the conditions (2), (4), (5), (6) and (7).

3. EDGE DISLOCATIONS IN THE COMPOSITE MEDIUM

In order to apply the dislocation array approach to the present problem, we first derive the complex potentials for the edge dislocation line running perpendicular to x, y plane through the point P ($x = s$, $y = 0$) on the crack line in the semi-infinite medium with surface layer (see Fig. 2).

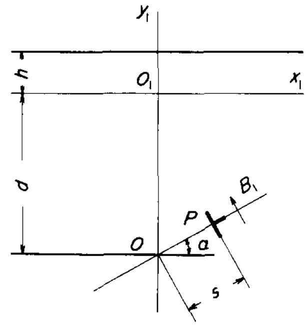

Fig. 2. Edge dislocation in a semi-infinite medium with a surface layer.

Since z_1 coordinate of P is $-id + se^{i\alpha}$, the complex potentials expressing the isolated edge dislocation at P in the *infinite* body are given by

$$\Delta\Phi_{m_1}^{(0)}(z_1) = -\frac{\mu_m B_1}{\pi(\kappa_m + 1)} \frac{e^{i\alpha}}{z_1 + id - se^{i\alpha}}$$

$$\Delta\Psi_{m_1}^{(0)}(z_1) = -\frac{\mu_m B_1}{\pi(\kappa_m + 1)} \left[\frac{e^{-i\alpha}}{z_1 + id - se^{i\alpha}} + \frac{(id + se^{-i\alpha})e^{i\alpha}}{(z_1 + id - se^{i\alpha})^2}\right] \tag{9}$$

where B_1 is the Burgers vector of the edge dislocation whose slip plane is perpendicular to the crack line. Starting with these potentials and taking account of the conditions (4), (5) and (6), we can get the required complex potentials for the edge dislocation at P in the composite semi-infinite medium in the form

$$\Delta\Phi_{m_1}^{(1)}(z_1) = -\frac{\mu_m B_1}{\pi(\kappa_m + 1)} \left\{\frac{e^{i\alpha}}{z_1 + id - se^{i\alpha}} - i\int_0^\infty a_1(m) \exp[-im(z_1 - id - se^{-i\alpha})]\, dm\right\}$$

$$\Delta\Psi_{m_1}^{(1)}(z_1) = -\frac{\mu_m B_1}{\pi(\kappa_m + 1)} \left\{\frac{e^{-i\alpha}}{z_1 + id - se^{i\alpha}} + \frac{(id + se^{-i\alpha})e^{i\alpha}}{(z_1 + id - se^{i\alpha})^2} - i\int_0^\infty [b_1(m) + imz_1 a_1(m)] \exp[-im(z_1 - id - se^{-i\alpha})]\, dm\right\}$$

$$\Delta\Phi_{s_1}^{(1)}(z_1) = i\frac{\mu_m B_1}{\pi(\kappa_m + 1)} \int_0^\infty \{c_1(m) \exp[im(z_1 + id - se^{i\alpha})] + d_1(m) \exp[-im(z_1 - id - se^{-i\alpha})]\}\, dm$$

$$\Delta\Psi_{s_1}^{(1)}(z_1) = i\frac{\mu_m B_1}{\pi(\kappa_m + 1)} \int_0^\infty \{[e_1(m) - imz_1 c_1(m)] \exp[im(z_1 + id - se^{i\alpha})] + [f_1(m) + imz_1 d_1(m)] \exp[-im(z_1 - id - se^{-i\alpha})]\}\, dm \tag{10}$$

where

$$a_1(m) = [e^{i\alpha}\Delta_1(m) - e^{-i\alpha}\Delta_2(m)]/\Delta(m)$$

$$b_1(m) = [e^{i\alpha}\Delta_3(m) + e^{-i\alpha}\Delta_4(m)]/\Delta(m)$$

$$c_1(m) = \delta e^{i\alpha} + (\eta - 1)[\overline{a_1(m)} + \overline{b_1(m)}]$$

$$d_1(m)=(\eta-1)[e^{i\alpha}+2m(d-s\sin\alpha)e^{-i\alpha}]+\delta a_1(m)$$

$$e_1(m)=-[\delta-\eta 2m(d-s\sin\alpha)]e^{i\alpha}+\eta e^{-i\alpha}+(\delta-\eta)\overline{a_1(m)}-(\eta-1)\overline{b_1(m)}$$

$$f_1(m)=-(\eta-1)e^{i\alpha}+[\delta-1-(\eta-1)2m(d-s\sin\alpha)]e^{-i\alpha}-(\delta-\eta)a_1(m)+\eta b_1(m) \quad (11)$$

$$\Delta(m)=\{\delta-[\delta-1+(\eta-1)2mh]e^{-2mh}\}[\eta-(\eta-1)e^{-2mh}]+\{\eta-\delta 2mh-(\eta-1)e^{-2mh}\}\times(\eta-1)2mhe^{-2mh}$$

$$\Delta_1(m)=(\eta-1)^2 4m^2h^2e^{-2mh}-[\eta-(\eta-1)e^{-2mh}](\eta-1-\eta e^{-2mh})$$

$$\Delta_2(m)=[\eta-(\eta-1)e^{-2mh}]\{(\eta-1-\eta e^{-2mh})2m(d-s\sin\alpha)-\delta 2mhe^{-2mh}\}+(\eta-1)2mhe^{-2mh}\{\delta-1-(\eta-1)4m^2h(d-s\sin\alpha)-\delta e^{-2mh}\} \quad (12)$$

$$\Delta_3(m)=\{\delta-[\delta-1+(\eta-1)2mh]e^{-2mh}\}(\eta-1)2mh+\{\eta-\delta 2mh-(\eta-1)e^{-2mh}\}\times(\eta-1-\eta e^{-2mh})$$

$$\Delta_4(m)=\{\eta-\delta 2mh-(\eta-1)e^{-2mh}\}\{(\eta-1-\eta e^{-2mh})2m(d-s\sin\alpha)-\delta 2mhe^{-2mh}\}-\{\delta-[\delta-1+(\eta-1)2mh]e^{-2mh}\}\{\delta-1-(\eta-1)4m^2h(d-s\sin\alpha)-\delta e^{-2mh}\}.$$

It is easily shown that the stresses corresponding to these potentials tend to zero for $|z_1|\to\infty$, so that they do not disturb the loading condition at infinity.

Similarly, we can obtain the complex potentials for the edge dislocation of Burgers vector B_2, which is located at the point P in the composite medium and whose slip plane is parallel to the crack line, as

$$\Delta\Phi^{(2)}_{m_1}(z_1)=i\frac{\mu_m B_2}{\pi(\kappa_m+1)}\left\{\frac{e^{i\alpha}}{z_1+id-se^{i\alpha}}+\int_0^\infty a_2(m)\exp[-im(z_1-id-se^{-i\alpha})]\,dm\right\}$$

$$\Delta\Psi^{(2)}_{m_1}(z_1)=i\frac{\mu_m B_2}{\pi(\kappa_m+1)}\left\{-\frac{e^{-i\alpha}}{z_1+id-se^{i\alpha}}+\frac{(id+se^{-i\alpha})e^{i\alpha}}{(z_1+id-se^{i\alpha})^2}+\int_0^\infty[b_2(m)+imz_1a_2(m)]\exp[-im(z_1-id-se^{-i\alpha})]\,dm\right\}$$

$$\Delta\Phi^{(2)}_{s_1}(z_1)=i\frac{\mu_m B_2}{\pi(\kappa_m+1)}\int_0^\infty\left\{c_2(m)\exp[im(z_1+id-se^{i\alpha})]+d_2(m)\exp[-im(z_1-id-se^{-i\alpha})]\right\}dm$$

$$\Delta\Psi^{(2)}_{s_1}(z_1)=i\frac{\mu_m B_2}{\pi(\kappa_m+1)}\int_0^\infty\{[e_2(m)-imz_1c_2(m)]\exp[im(z_1+id-se^{i\alpha})]+[f_2(m)+imz_1d_2(m)]\exp[-im(z_1-id-se^{-i\alpha})]\}\,dm \quad (13)$$

where

$$a_2(m)=-i[e^{i\alpha}\Delta_1(m)+e^{-i\alpha}\Delta_2(m)]/\Delta(m)$$

$$b_2(m)=-i[e^{i\alpha}\Delta_3(m)-e^{-i\alpha}\Delta_4(m)]/\Delta(m)$$

$$c_2(m)=-i\delta e^{i\alpha}+(\eta-1)[\overline{a_2(m)}+\overline{b_2(m)}]$$

$$d_2(m)=-i(\eta-1)[e^{i\alpha}-2m(d-s\sin\alpha)e^{-i\alpha}]+\delta a_2(m) \quad (14)$$

$$e_2(m)=i[\delta-\eta 2m(d-s\sin\alpha)]e^{i\alpha}+i\eta e^{-i\alpha}+(\delta-\eta)\overline{a_2(m)}-(\eta-1)\overline{b_2(m)}$$

$$f_2(m)=i(\eta-1)e^{i\alpha}+i[\delta-1-(\eta-1)2m(d-s\sin\alpha)]e^{-i\alpha}-(\delta-\eta)a_2(m)+\eta b_2(m).$$

By superposing these potentials, we get the complex potentials expressing the continuous distribution of edge dislocations on the crack line L in the composite medium. Denoting Burgers vector of dislocations on the line element ds at the point P by $B_1(s)\,ds$ and $B_2(s)\,ds$ respectively, from eqns (10) and (13) we have

$$\Phi^{(1)}_{m_1}(z_1) = -\frac{\mu_m}{\pi(\kappa_m+1)}\int_{-a}^{a} B_1(s)\,ds\left\{\frac{e^{i\alpha}}{z_1+id-se^{i\alpha}} - i\int_0^\infty a_1(m)\exp\left[-im(z_1-id-se^{-i\alpha})\right]dm\right\}$$

$$\Psi^{(1)}_{m_1}(z_1) = -\frac{\mu_m}{\pi(\kappa_m+1)}\int_{-a}^{a} B_1(s)\,ds\left\{\frac{e^{-i\alpha}}{z_1+id-se^{i\alpha}} + \frac{(id+se^{-i\alpha})e^{i\alpha}}{(z_1+id-se^{i\alpha})^2}\right.$$
$$\left. - i\int_0^\infty \left[b_1(m)+imz_1a_1(m)\right]\exp\left[-im(z_1-id-se^{-i\alpha})\right]dm\right\} \qquad (15)$$

$$\Phi^{(2)}_{m_1}(z_1) = i\frac{\mu_m}{\pi(\kappa_m+1)}\int_{-a}^{a} B_2(s)\,ds\left\{\frac{e^{i\alpha}}{z_1+id-se^{i\alpha}} + \int_0^\infty a_2(m)\exp\left[-im(z_1-id-se^{-i\alpha})\right]dm\right\}$$

$$\Psi^{(2)}_{m_1}(z_1) = i\frac{\mu_m}{\pi(\kappa_m+1)}\int_{-a}^{a} B_2(s)\,ds\left\{-\frac{e^{-i\alpha}}{z_1+id-se^{i\alpha}} + \frac{(id+se^{-i\alpha})e^{i\alpha}}{(z_1+id-se^{i\alpha})^2}\right.$$
$$\left. + \int_0^\infty \left[b_2(m)+imz_1a_2(m)\right]\exp\left[-im(z_1-id-se^{-i\alpha})\right]dm\right\} \qquad (16)$$

and corresponding potentials $\Phi^{(1)}_{s_1}(z_1)$, $\Psi^{(1)}_{s_1}(z_1)$ and $\Phi^{(2)}_{s_1}(z_1)$, $\Psi^{(2)}_{s_1}(z_1)$ for the surface layer, whose expressions are omitted here for brevity. It is evident that these potentials satisfy the boundary conditions (4), (5) and (6) and corresponding stresses tend to zero for $|z_1|\to\infty$. The physical condition of single-valuedness of displacement vector due to the dislocation arrays on L yeilds

$$\int_{-a}^{a} B_i(s)\,ds = 0. \quad (i=1,2). \qquad (17)$$

4. SINGULAR INTEGRAL EQUATIONS FOR DISLOCATION DENSITY

Now we will proceed to the crack problem. Let us consider the following complex potentials

$$\Phi_{m_1}(z_1) = \tfrac{1}{4}\sigma_0 + \Phi^{(1)}_{m_1}(z_1) + \Phi^{(2)}_{m_1}(z_1), \quad \Psi_{m_1}(z_1) = -\tfrac{1}{2}\sigma_0 + \Psi^{(1)}_{m_1}(z_1) + \Psi^{(2)}_{m_1}(z_1),$$
$$\Phi_{s_1}(z_1) = \frac{\delta+\eta-1}{4}\sigma_0 + \Phi^{(1)}_{s_1}(z_1) + \Phi^{(2)}_{s_1}(z_1), \quad \Psi_{s_1}(z_1) = -\frac{\delta+\eta-1}{2}\sigma_0 + \Psi^{(1)}_{s_1}(z_1) + \Psi^{(2)}_{s_1}(z_1). \qquad (18)$$

It is evident that the boundary conditions (2), (4)–(6) are satisfied by the above. Hence the problem is reduced to the determination of dislocation density $B_1(s)$ and $B_2(s)$ by the traction free condition (7) on the rim of the crack. To this end it is convenient to express the complex potentials for the semi-infinite medium in terms of z with the aid of formulas (8). Then substituting them into eqn (7), we get the following set of singular integral equations

$$\frac{1}{\pi}\int_{-1}^{1}\frac{F_1(S)\,dS}{S-X} + \frac{1}{2\pi}\int_{-1}^{1}\left[K_1(S,X)F_1(S) - K_4(S,X)F_2(S)\right]dS = 1-\cos 2\alpha$$

$$(|X|<1)$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{F_2(S)\,dS}{S-X} - \frac{1}{2\pi}\int_{-1}^{1}\left[K_2(S,X)F_1(S) + K_3(S,X)F_2(S)\right]dS = -\sin 2\alpha \qquad (19)$$

where the first integrals are understood to be the Cauchy principal value and

$$K_1(S,X) + iK_2(S,X) = i\int_0^\infty a_{10}(\xi)\exp\left[-i\xi(Xe^{i\alpha} - i2D - Se^{-i\alpha})\right]d\xi$$

$$- i\int_0^\infty \overline{a_{10}(\xi)} \exp\,[i\xi(Xe^{-i\alpha} + i2D - Se^{i\alpha})]\,\mathrm{d}\xi$$

$$- ie^{-i2\alpha}\int_0^\infty \overline{[b_{10}(\xi)}$$

$$+ i\xi\{X(e^{i\alpha} - e^{-i\alpha} - i2D\}\overline{a_{10}(\xi)}] \exp\,[i\xi(Xe^{-i\alpha} + i2D - Se^{i\alpha})]\,\mathrm{d}\xi$$

$$K_3(S, X) + iK_4(S, X) = \int_0^\infty a_{20}(\xi) \exp\,[-i\xi(Xe^{i\alpha} - i2D - Se^{-i\alpha})]\,\mathrm{d}\xi$$

$$- \int_0^\infty \overline{a_{20}(\xi)} \exp\,[i\xi(Xe^{-i\alpha} + i2D - Se^{i\alpha})]\,\mathrm{d}\xi$$

$$- e^{-i2\alpha}\int_0^\infty \overline{[b_{20}(\xi)}$$

$$+ i\xi\{X(e^{i\alpha} - e^{-i\alpha}) - i2D\}\overline{a_{20}(\xi)}] \exp\,[i\xi(Xe^{-i\alpha} + i2D - Se^{i\alpha})]\,\mathrm{d}\xi. \tag{20}$$

Here the following substitution is made for normalization

$$S = s/a, \quad X = x/a, \quad D = d/a, \quad H = h/a, \quad \xi = ma, \quad F_i(S) = -\frac{4\mu_m}{(\kappa_m + 1)\sigma_0} B_i(s),$$

$$a_{i0}(\xi; H, D, S) = a_i(m; h, d, s), \quad b_{i0}(\xi; H, D, S) = b_i(m; h, d, s). \quad (i = 1, 2) \tag{21}$$

By substitution of eqn (21), the single-valuedness condition (17) becomes

$$\int_{-1}^{1} F_i(S)\,\mathrm{d}S = 0. \qquad (i = 1, 2) \tag{22}$$

The solution of the system of singular integral eqn (19) under the conditions (22) has already been discussed by Erdogan[10]. Following his technique, we assume the unknown functions in the form

$$F_i(S) = (1 - S^2)^{-1/2}\left[A_0^{(i)} + \sum_{n=1}^{\infty} A_n^{(i)} T_n(S)\right] \quad (i = 1, 2) \tag{23}$$

where $A_n^{(i)}$ are unknown constants and $T_n(S)$ are Chebyshev polynomials of the first kind. Substituting the above into eqn (22) and integrating, we obtain

$$A_0^{(i)} = 0. \quad (i = 1, 2) \tag{24}$$

Substituting eqn (23) into eqn (19) and taking account of the orthogonality relations of Chebyshev polynomials[13], we finally have the following set of linear equations for the unknown constants

$$\sum_{n=1}^{\infty} [(\delta_{kn} + c_{kn}^{(1)})A_n^{(1)} - c_{kn}^{(4)}A_n^{(2)}] = (1 - \cos 2\alpha)\delta_{1k}$$

$$(k = 1, 2, 3, \ldots) \tag{25}$$

$$\sum_{n=1}^{\infty} [-c_{kn}^{(2)}A_n^{(1)} + (\delta_{kn} - c_{kn}^{(3)})A_n^{(2)}] = -\sin 2\alpha\delta_{1k}$$

where δ_{kn} is Kronecker delta and

$$c_{kn}^{(j)} = \frac{1}{\pi^2}\int_{-1}^{1} U_{k-1}(X)\sqrt{(1 - X^2)}\left[\int_{-1}^{1} K_j(S, X)\frac{T_n(S)}{\sqrt{(1 - S^2)}}\,\mathrm{d}S\right]\mathrm{d}X. \quad (j = 1, 2, 3, 4) \tag{26}$$

$U_n(X)$ are Chebyshev polynomials of the second kind. All the integrals in eqn (26) are of Gauss–Chebyshev type and may easily be evaluated by using proper quadrature formulas[14].

5. STRESS INTENSITY FACTOR

In the application of the results to fracture problems in composite materials, of particular interest are the stress intensity factors and the probable angle of crack growth at crack tips.

It is easily shown that in complex potentials in eqn (18), following terms contribute most importantly to the singular behavior of stresses near the ends of the crack

$$\Phi_m(z) = \frac{\mu_m}{\pi(\kappa_m+1)} \int_{-a}^{a} \frac{B_1(s) - iB_2(s)}{s-z}\,\mathrm{d}s$$

$$\Psi_m(z) = \frac{\mu_m}{\pi(\kappa_m+1)} \left\{ \int_{-a}^{a} \frac{B_1(s) + iB_2(s)}{s-z}\,\mathrm{d}s - \int_{-a}^{a} \frac{s[B_1(s) - iB_2(s)]}{(s-z)^2}\,\mathrm{d}s \right\}. \tag{27}$$

By the theorem on the behavior of Cauchy integrals in the neighborhood of the ends of the path of integration[15] and after some manipulations, we obtain the expression of the ordinarily defined complex stress intensity factor as

$$k_{1A} + ik_{2A} = \frac{\sigma_0\sqrt{a}}{2} \sum_{n=1}^{\infty} (A_n^{(1)} - iA_n^{(2)}),$$

$$k_{1B} + ik_{2B} = \frac{\sigma_0\sqrt{a}}{2} \sum_{n=1}^{\infty} (-1)^{n+1}(A_n^{(1)} - iA_n^{(2)}), \tag{28}$$

where the subscripts A and B correspond to the crack tips $A(z = a)$ and $B(z = -a)$ respectively. Once the set of algebraic eqns (25) is solved for $A_n^{(i)}$, stress intensity factors are easily evaluated by the above equations.

The normal and shear components, k_1 and k_2, are a measure of the intensity of the stresses around the crack tip. For instance, the cleavage stress at the crack tip A may be expressed as[11]

$$\tau_{\phi\phi} = \frac{1}{4\sqrt{(2r)}} \left[k_{1A}\left(3\cos\frac{\phi}{2} + \cos\frac{3\phi}{2}\right) - k_{2A}\left(3\sin\frac{\phi}{2} + 3\sin\frac{3\phi}{2}\right) \right], \tag{29}$$

where (r, ϕ) are the polar coordinates at the crack tip A, ϕ being measured from the prolongation of the crack. Based on Erdogan and Sih criterion[11] on the plane of cleavage for brittle solids, the probable angle ϕ_0 of crack growth may be postulated as the angle (of the radial plane) corresponding to the maximum cleavage stress and thus may be determined from

$$k_{2A}(1 - 3\cos\phi_0) - k_{1A}\sin\phi_0 = 0. \tag{30}$$

Hence the angle ϕ_0 depends on the crack orientation α through the stress intensity factors ratio k_{1A}/k_{2A}.

6. NUMERICAL RESULTS

In order to show the effect of the surface layer, some numerical calculations are carried out. Once the geometrical configuration parameters $(h/d, a/d, \alpha)$ and materials combination (δ, η) are specified, the system of linear eqns (25) is solved by an approximate method in which only the first N equations containing only the first N unknowns are taken. In every case, we take the equivalent Poisson's ratio as $\kappa_m = \kappa_s = 2$.

In the calculated results, it is found that the value of N needed to achieve a particular level of accuracy is strongly dependent on a/d. An example is shown in Table 1 which corresponds to the simple case of a crack perpendicular to the surface of a semi-infinite medium without a surface layer. This table manifests the values of N which is required for $k_{1A}/\sigma_0\sqrt{a}$ to be correct to the decimal places, shown in the table, for various values of a/d. Isida's results[16], due to the Laurent series expansion technique and obtained from the polynomials up to $(a/d)^{71}$ are also

Table 1. Stress intensity factors for a crack perpendicular to the free surface of a semi-infinite medium

a/d		0.60	0.70	0.80	0.90	0.95
$k_{IA}/\sigma_0\sqrt{a}$	Isida	1.148 97	1.237 85	1.387 5	1.706 4	2.078
	eq. (28)	1.148 97	1.237 85	1.387 5	1.707 7	2.121
$k_{IB}/\sigma_0\sqrt{a}$	Isida	1.077 63	1.107 44	1.146 4	1.203 8	1.250
	eq. (28)	1.077 63	1.107 44	1.146 4	1.203 8	1.251
N		12	13	15	20	30

shown for comparison. It may be understood that the approximation mentioned above gives satisfactory results even in the case where the crack tip is quite close to the free surface. The computation also reveals that the convergence of the series (28) is fairly good in the more general combinations of parameters.

Figures 3–5 show the effect of the width ratio h/d of the surface layer, where the normal and shear components of stress intensity factor are plotted versus the crack angle α for three cases of width ratio, $h/d = 0{\cdot}01$, $0{\cdot}1$ and $1{\cdot}0$. Here the length of the crack (a/d) and the shear modulus ratio Γ are fixed as $a/d = 0{\cdot}7$ and $\Gamma = 3$. In the case of thin surface layer $(h/d = 0{\cdot}01)$, the normal

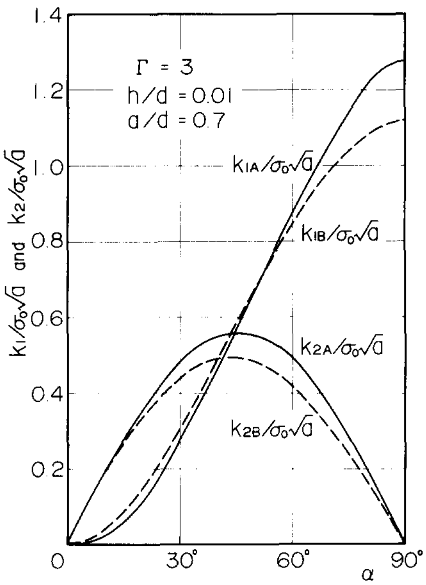

Fig. 3. Stress intensity factors vs crack orientation α ($\Gamma = 3$, $h/d = 0{\cdot}01$, $a/d = 0{\cdot}7$, $\kappa_m = \kappa_s = 2$).

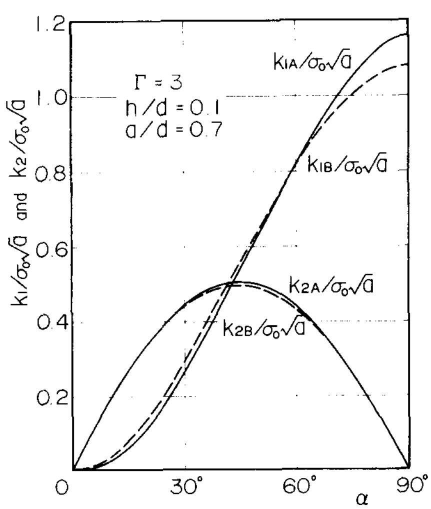

Fig. 4. Stress intensity factors vs crack orientation α ($\Gamma = 3$, $h/d = 0{\cdot}1$, $a/d = 0{\cdot}7$, $\kappa_m = \kappa_s = 2$).

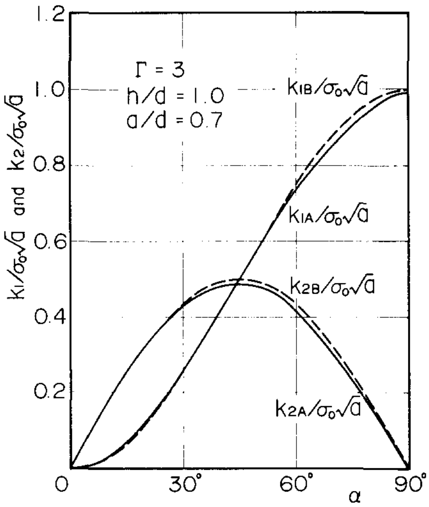

Fig. 5. Stress intensity factors vs crack orientation α ($\Gamma = 3$, $h/d = 1{\cdot}0$, $a/d = 0{\cdot}7$, $\kappa_m = \kappa_s = 2$).

component of stress intensity factor at the crack tip near the surface layer (k_{1A}) becomes larger than that at the other crack tip (k_{1B}) when α is about 90°. If the surface layer becomes thicker ($h/d = 0{\cdot}1$, $1{\cdot}0$), k_{1A} and k_{1B} become smaller and in the case of $h/d = 1{\cdot}0$ the order is reversed when α approaches to 90°. Figure 6 corresponds to the case of $\Gamma = 10$, $h/d = 1{\cdot}0$ and $a/d = 0{\cdot}7$. Comparing with Fig. 5, we see that the effect of the surface layer increases with the increase of the shear modulus ratio. If k_{1A} and k_{1B} are equal to $\sigma_0\sqrt{a}$ for some values of Γ and h in the case of $\alpha = 90°$, the function of that surface layer may be said to be equivalent to the semi-infinite body of the same material as the matrix, since $\sigma_0\sqrt{a}$ is the stress intensity factor at the crack tip in the infinite medium subjected to the tensile load σ_0 which is perpendicular to the crack of length $2a$. Such a surface layer may be called the equivalent stiffner. Figure 7 shows an example of the equivalent stiffner for the case of $a/d = 0{\cdot}7$ and $\alpha = \pi/2$.

Figures 8 and 9 show the variation of k_{1A} and k_{1B} due to a/d for $h/d = 0{\cdot}01$ and $1{\cdot}0$ respectively, Γ being 5. It is readily seen that the effect of surface layer becomes remarkable when the crack tip comes nearer to the surface layer.

By using the values of k_1 and k_2, we are able to obtain the probable angles of crack growth ϕ_0. In Tables 2 and 3 the results for the crack tip A are shown for some combinations of parameters. Here the case of $a/d = 0$ corresponds to that of a single crack in an infinite medium. It may be seen that the variation of ϕ_0 due to a/d is not so remarkable.

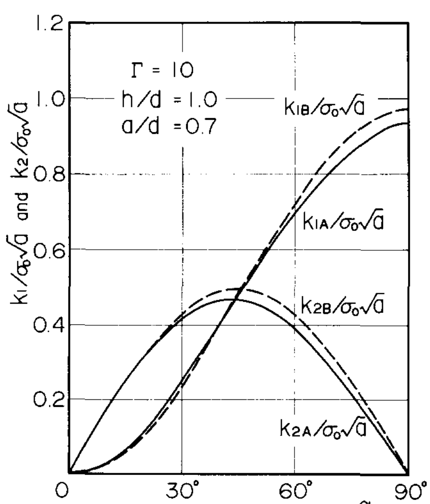

Fig. 6. Stress intensity factors vs crack orientation α ($\Gamma = 10$, $h/d = 1{\cdot}0$, $a/d = 0{\cdot}7$, $\kappa_m = \kappa_s = 2$).

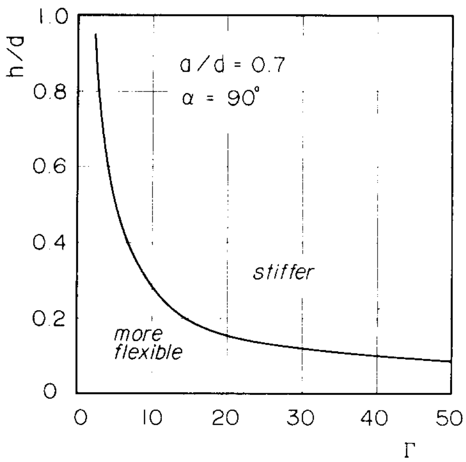

Fig. 7. Width ratio h/d vs Γ for equivalent stiffner ($a/d = 0{\cdot}7$, $\alpha = 90°$, $\kappa_m = \kappa_s = 2$).

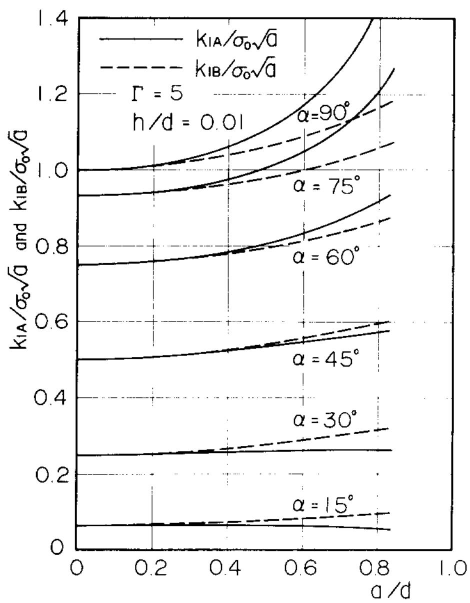

Fig. 8. Stress intensity factors vs crack length ratio a/d ($\Gamma = 5$, $h/d = 0{\cdot}01$, $\kappa_m = \kappa_s = 2$).

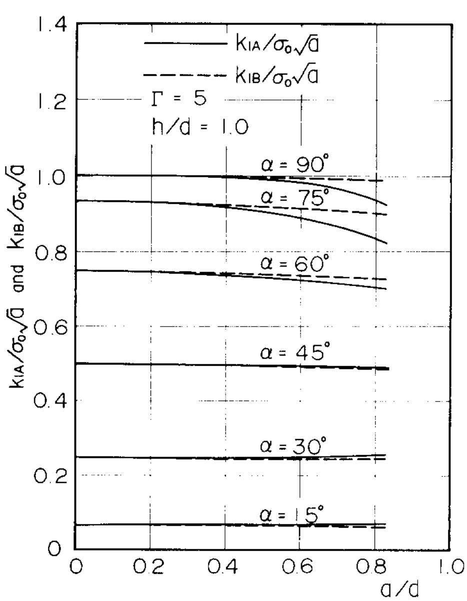

Fig. 9. Stress intensity factors vs crack length ratio a/d ($\Gamma = 5$, $h/d = 1{\cdot}0$, $\kappa_m = \kappa_s = 2$).

Table 2. Probable crack growth angles ϕ_0 for crack orientation and crack length ($\Gamma = 3$)

$h/d = 0.01$

	a/d						
α	0	0.3	0.4	0.5	0.6	0.7	0.8
15°	65.5°	65.6°	65.7°	65.8°	66.0°	66.3°	66.6°
30°	60.0°	59.9°	60.0°	60.1°	60.2°	60.4°	60.6°
45°	53.1°	53.0°	52.9°	52.9°	53.0°	53.1°	53.3°
60°	43.2°	42.9°	42.8°	42.7°	42.6°	42.7°	42.9°
75°	26.7°	26.4°	26.2°	26.0°	25.9°	26.0°	26.2°

$h/d = 1.0$

	a/d						
α	0	0.3	0.4	0.5	0.6	0.7	0.8
15°	65.5°	65.5°	65.5°	65.4°	65.4°	65.4°	65.3°
30°	60.0°	60.0°	59.9°	59.9°	59.8°	59.8°	59.7°
45°	53.1°	53.1°	53.1°	53.0°	52.9°	52.8°	52.6°
60°	43.2°	43.2°	43.2°	43.1°	43.0°	42.7°	42.4°
75°	26.7°	26.7°	26.7°	26.6°	26.5°	26.2°	25.7°

Table 3. Probable crack growth angles ϕ_0 for crack orientation and crack length ($\Gamma = 10$)

$h/d = 0.01$

	a/d						
α	0	0.3	0.4	0.5	0.6	0.7	0.8
15°	65.5°	65.6°	65.6°	65.8°	66.0°	66.2°	66.5°
30°	60.0°	60.0°	60.0°	60.1°	60.2°	60.4°	60.6°
45°	53.1°	53.0°	52.9°	52.9°	53.0°	53.1°	53.3°
60°	43.2°	42.9°	42.8°	42.7°	42.7°	42.8°	43.0°
75°	26.7°	26.4°	26.3°	26.1°	26.0°	26.1°	26.3°

$h/d = 1.0$

	a/d						
α	0	0.3	0.4	0.5	0.6	0.7	0.8
15°	65.5°	65.5°	65.4°	65.3°	65.2°	65.1°	65.0°
30°	60.0°	60.0°	59.9°	59.8°	59.7°	59.6°	59.4°
45°	53.1°	53.1°	53.1°	53.0°	52.8°	52.6°	52.3°
60°	43.2°	43.3°	43.2°	43.1°	42.9°	42.5°	41.8°
75°	26.7°	26.8°	26.8°	26.7°	26.5°	26.0°	25.0°

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